UNIQUENESS OF SPACELIKE HYPERSURFACES WITH CONSTANT HIGHER ORDER MEAN CURVATURE IN GENERALIZED ROBERTSON-WALKER SPACETIMES

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ABSTRACT. In this paper we study the problem of uniqueness for spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker (GRW) spacetimes. In particular, we consider the following question: Under what conditions must a compact spacelike hypersurface with constant higher order mean curvature in a spatially closed GRW spacetime be a spacelike slice ? We prove that this happens, esentially, under the so called *null convergence condition*. Our approach is based on the use of the Newton transformations (and their associated differential operators) and the Minkowski formulae for spacelike hypersurfaces.

1. INTRODUCTION

The mathematical interest for the study of spacelike hypersurfaces in spacetimes began in the seventies, with the works of Calabi [13], Cheng and Yau [15], Brill and Flaherty [12], Choquet-Bruhat [16, 17], and later on with some other authors (cf. e.g. [23, 33, 11, 22, 9, 10]). Moreover, the study of such hypersurfaces is also of interest from a physical point of view, because of its relation to several problems in general relativity (see, for instance, [18, 19, 26]). A basic question on this subject is the uniqueness of spacelike hypersurfaces with constant mean curvature in certain spacetimes (we refer the reader to the introductions of the papers [4] and [2] for a brief account of it). More recently, there has been also an increasing interest in the study of uniqueness of spacelike hypersurfaces with constant higher order mean curvature, including the case of the second order mean curvature which is closely related to the intrinsic scalar curvature of the hypersurface (see Section 2).

Conformally stationary (CS) spacetimes are time-orientable spacetimes which are equipped with a globally defined timelike conformal vector field, and they include, for instance, the family of generalized Robertson-Walker (GRW) spacetimes. By a GRW spacetime, we mean a Lorentzian warped product $-I \times_f M^n$ with Riemannian fibre M^n and warping function f. In particular, when the Riemannian factor M^n has constant sectional curvature then $-I \times_f M^n$ is classically called a Robertson-Walker (RW) spacetime (for the details, see Section 2). In a GRW spacetime, the vector field given by $K(t,x) = f(t)(\partial/\partial_t)_{(t,x)}$ defines globally a timelike conformal field, which is also closed, in the sense that its metrically equivalent 1-form is closed. As observed by Montiel in [28], every CS spacetime which

is equipped with a closed timelike conformal vector field is locally isometric to a GRW.

In this paper, we are interested in the study of uniqueness of compact spacelike hypersurfaces with constant higher order mean curvature in GRW spacetimes. First of all, recall that if a GRW spacetime $-I \times_f M^n$ admits a compact spacelike hypersurface, then it must be spatially closed, in the sense that the Riemannian factor M^n must be compact [4, Proposition 3.2 (i)]. On the other hand, observe that, for a spatially closed GRW spacetime $-I \times_f M^n$, the family of slices $M_t^n = \{t\} \times M^n$ constitutes a foliation of $-I \times_f M^n$ by compact totally umbilical leaves with constant mean curvature $H(t) = f'(t)/f(t)$ and, more generally, constant k-th mean curvature $H_k(t) = (f'(t)/f(t))^k$ (for the details, see [4]). Therefore, it is natural to address the following question:

> *Under what conditions must a compact spacelike hypersurface with constant higher order mean curvature in a spatially closed GRW spacetime be a spacelike slice $M_t^n = \{t\} \times M^n$?*

In [4], the first author, jointly with Romero and Sánchez, considered that question for the case of constant mean curvature hypersurfaces. In particular, they found that, when the ambient space obeys the so called *timelike convergence condition*, then every compact spacelike hypersurface with constant mean curvature must be totally umbilical and, except in very exceptional cases, it must be a spacelike slice. As observed by those same authors in [5], the weaker *null convergence condition* is enough to guarantee that the hypersurface must be totally umbilical, not only for hypersurfaces into GRW spacetimes but, more generally, for hypersurfaces into CS spacetimes equipped with a timelike conformal vector field which is an eigenfield of the Ricci operator (and, in particular, for CS spacetimes which are equipped with a closed timelike conformal vector field). Later on, Montiel [28] considered again that question and, after a careful classification of the totally umbilical hypersurfaces with constant mean curvature, he completed the program by showing that the only compact spacelike hypersurfaces with constant mean curvature into a GRW spacetime which satisfies the null convergence condition are the spacelike slices, unless in the case where the spacetime is a de Sitter space and the hypersurface is a round umbilical sphere.

With respect to the case of hypersurfaces with constant higher order mean curvature, in [28] Montiel also obtained a uniqueness result for hypersurfaces with constant scalar curvature (equivalently, constant second order mean curvature) into CS spacetimes with constant sectional curvature. More recently, the authors, jointly with Brasil Jr., initiated the study of hypersurfaces with constant higher order mean curvature in CS spacetimes [2].

In all the references above considering that question, the main tool used to extract information about the spacelike hypersurfaces and prove the results are the so called Minkowski formulae. Actually, the use of Minkowski-type integral formulae in this context was first started by Montiel in [27] for the study of hypersurfaces with constant mean curvature in de Sitter space, and it was continued later by Alías, Romero and Sánchez in [4, 5, 6] for constant mean curvature hypersurfaces in GRW spacetimes and, more generally, in CS spacetimes. Let us remark here that for the case of the mean curvature, only the first and second Minkowski formulae come into play. However, for the case of higher order mean curvatures, one needs to use successive Minkowski formulae, which involve covariant derivatives

of the Ricci tensor of the ambient spacetime. That makes the task much harder, unless one assumes that the ambient spacetime is Einstein or, even more, it has constant sectional curvature. That is the case, for instance, in [1] where higher order Minkowski formulae are developed for spacelike hypersurfaces in de Sitter, in [28] where the third Minkowski formula is written for spacelike hypersurfaces in a CS spacetime with constant sectional curvature, and, more generally, in [2] where the authors, jointly with Brasil Jr., derived general Minkowski formulae for spacelike hypersurfaces in CS spacetimes with constant sectional curvature.

In this paper, which is a natural continuation of the references above, we go deeper into the study of spacelike hypersurfaces with constant higher order mean curvature in spatially closed GRW spacetimes. Our approach, which is based in the use of the so called Newton transformations P_k and their associated second order differential operators L_k (see Section 3), allows us to extend the previous uniqueness results to the case where the ambient spacetimes do not have constant sectional curvature. For instance, and as a first application, in Section 5 we obtain the following (Theorem 5.1):

> Let $-I \times_f M^n$ a spatially closed GRW spacetime such that its warp-ing function satisfies
>
> $$f f'' - f'^2 \leq 0,$$
>
> (that is, such that $-\log f$ is convex). Let Σ^n be a compact spacelike hypersurface immersed into $-I \times_f M^n$ whose k-th Newton trans-formation P_k is definite on Σ, for some $k = 0, 1 \ldots, n-1$. If the quotient H_{k+1}/H_k is constant, then the hypersurface is an embed-ded slice $\{t_0\} \times M$, where $t_0 \in I$ satisfies $f'(t_0) \neq 0$ if $k \geq 1$.

The proof of Theorem 5.1 is based strongly on the ellipticity of the differential operator L_k (or, equivalently, the definiteness of P_k; for the details, see Section 3). In Corollary 5.2 and Corollary 5.4 we apply the result above to situations where the ellipticity of L_k can be derived from geometric hypothesis.

On the other hand, in Section 6 we derive general Minkowski formulae for hypersurfaces in GRW spacetimes which can be applied to the study of hypersurfaces with constant higher order mean curvature in arbitrary RW spacetimes, even if the ambient spacetime does not have constant sectional curvature. Our derivation of Minkowski formulae in this generality is based on the use of the operators P_k and L_k, and on the detailed analysis of a series of terms which come into play when the ambient space does not have constant sectional curvature. In particular, we are able to find a neat expression for the divergence of P_k (see formula (43)) which allows us to write our Minkowski formulae in a nice way.

As a first application of these Minkowski formulae, in Section 7 we obtain the following uniqueness result (Theorem 7.2) under the assumption of null conver-gence condition (let us recall that a spacetime obeys the null convergence condition if its Ricci curvature is non-negative on null (or lightlike) directions; obviously, every spacetime with constant sectional curvature trivially obeys null convergence condition):

> Let $-I \times_f M^n$ be a spatially closed RW spacetime obeying the null convergence condition, with $n \geq 3$, that is
>
> (1) $$\kappa \geq \sup_I (f f'' - f'^2),$$

> *where κ is the constant sectional curvature of M^n. Then every compact spacelike hypersurface immersed into $-I \times_f M^n$ with positive constant H_2 is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if the inequality in (1) is strict.*

See also Theorem 7.3 and Theorem 7.4 for two different versions of Theorem 7.2 for the general case of hypersurfaces with constant higher order mean curvature H_k, when $k \geq 3$.

Finally, in Section 9 and as another application of the second order differential operators associated to the Newton transformations, we obtain the following uniqueness result for hypersurfaces in GRW spacetimes (Theorem 9.2):

> *Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the strong null convergence condition, with $n \geq 3$, that is,*

(2) $$K_M \geq \sup_I (f f'' - f'^2),$$

> *where K_M stands for the sectional curvature of M^n. Assume that Σ^n is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which is contained in a slab*
>
> $$\Omega(t_1, t_2) = (t_1, t_2) \times M$$
>
> *on which f' does not vanish. If H_k is constant, with $2 \leq k \leq n$ then Σ is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if we assume that the inequality in (2) is strict.*

The proof of Theorem 9.2 is based strongly on the ellipticity of the differential operators associated to the Newton transformations.

2. Preliminaries

Consider M^n an n-dimensional Riemannian manifold, and let I be a 1-dimensional manifold (either a circle or an open interval of $\mathbb{R}$). We will denote by $-I \times_f M^n$ the $(n+1)$-dimensional product manifold $I \times M$ endowed with the Lorentzian metric

$$\langle,\rangle = -dt^2 + f^2(t)\langle,\rangle_M,$$

where $f > 0$ is a positive smooth function on I, and $\langle,\rangle_M$ stands for the Riemannian metric on M^n. That is, $-I \times_f M^n$ is nothing but a Lorentzian warped product with Lorentzian base $(I, -dt^2)$, Riemannian fiber $(M^n, \langle,\rangle_M)$, and warping function f. Following the terminology introduced in [4], we will refer to $-I \times_f M^n$ as a generalized Robertson-Walker (GRW) spacetime. In particular, when the Riemannian factor M^n has constant sectional curvature then $-I \times_f M^n$ is classically called a Robertson-Walker (RW) spacetime.

Consider a smooth immersion $\psi : \Sigma^n \to -I \times_f M^n$ of an n-dimensional connected manifold Σ into a GRW spacetime, and assume that the induced metric via

ψ is a Riemannian metric on Σ; that is, Σ is a spacelike hypersurface. In that case, since

$$\partial_t = (\partial/\partial_t)_{(t,x)}, \quad (t,x) \in -I \times_f M^n,$$

is a unitary timelike vector field globally defined on the ambient GRW spacetime, then there exists a unique unitary timelike normal field N globally defined on Σ which is in the same time-orientation as ∂_t, so that

$$\langle N, \partial_t \rangle \leq -1 < 0 \quad \text{on} \quad \Sigma.$$

We will refer to that normal field N as the future-pointing Gauss map of the hypersurface. Its opposite will be refered as the past-pointing Gauss map of Σ.

Let $A : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ stand for the shape operator (or Weingarten endomorphism) of Σ with respect to either the future or the past-pointing Gauss map N. As is well known, A defines a self-adjoint linear operator on each tangent space $T_p\Sigma$, and its eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants given by

$$S_k(p) = \sigma_k(\kappa_1(p), \ldots, \kappa_n(p)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_k} x_{i_1} \ldots x_{i_k}.$$

Observe that the characteristic polynomial of A can be writen in terms of the S_k's as

$$(3) \qquad \det(tI - A) = \sum_{k=0}^{n} (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. The kth -mean curvature H_k of the hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k S_k = \sigma_k(-\kappa_1, \ldots, -\kappa_n),$$

for every $0 \leq k \leq n$. In particular, when $k = 1$

$$H_1 = -\frac{1}{n}\sum_{i=1}^{n} \kappa_i = -\frac{1}{n}\text{trace}(A) = H$$

is nothing but the mean curvature of Σ, which is the main extrinsic curvature of the hypersurface. The choice of the sign $(-1)^k$ in our definition of H_k is motivated by the fact that in that case the mean curvature vector is given by $\overrightarrow{H} = HN$. Therefore, $H(p) > 0$ at a point $p \in \Sigma$ if and only if $\overrightarrow{H}(p)$ is in the same time-orientation as N. Obviously, when k is even the sign of H_k does not depend on the chosen Gauss map. Even more, when k is even it follows from the Gauss equation of the hypersurface that the value of H_k is a geometric quantity which is related to the intrinsic curvature of Σ. For instance, for $k = 2$ one gets that

$$(4) \qquad n(n-1)H_2 = \bar{S} - S + 2\overline{\text{Ric}}(N, N),$$

where S and $\bar{S}$ are, respectively, the scalar curvature of Σ and $-I \times_f M^n$, and $\overline{\text{Ric}}$ stands for the Ricci tensor of the ambient GRW spacetime.

3. The Newton transformations and their associated differential operators

The classical Newton transformations $P_k : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ are defined inductively from A by

$$P_0 = I \quad \text{and} \quad P_k = \binom{n}{k} H_k I + A \circ P_{k-1},$$

for every $k = 1\ldots,n$, where I denotes the identity in $\mathcal{X}(\Sigma)$. Equivalently,

$$P_k = \sum_{j=0}^{k} \binom{n}{j} H_j A^{k-j}.$$

Note that by the Cayley-Hamilton theorem, we have $P_n = 0$ from (3). Observe also that when k is even, the definition of P_k does not depend on the choosen Gauss map, but when k is odd there is a change of sign in the definition of P_k.

Let us recall that each $P_k(p)$ is also a self-adjoint linear operator on each tangent space $T_p\Sigma$ which commutes with $A(p)$. Indeed, $A(p)$ and $P_k(p)$ can be simultaneously diagonalized: if $\{e_1, \ldots, e_n\}$ are the eigenvectors of $A(p)$ corresponding to the eigenvalues $\kappa_1(p), \ldots, \kappa_n(p)$, respectively, then they are also the eigenvectors of $P_k(p)$ with corresponding eigenvalues given by

$$\mu_{i,k}(p) = (-1)^k \frac{\partial \sigma_{k+1}}{\partial x_i}(\kappa_1(p), \ldots, \kappa_n(p)) = (-1)^k \sum_{i_1 < \cdots < i_k, i_j \neq i} \kappa_{i_1}(p) \cdots \kappa_{i_k}(p),$$

for every $1 \leq i \leq n$. From here it can be easily seen that

$$\text{(5)} \qquad\qquad\qquad\qquad \text{trace}(P_k) = c_k H_k,$$

$$\text{(6)} \qquad\qquad\qquad\qquad \text{trace}(A \circ P_k) = -c_k H_{k+1},$$

and

$$\text{(7)} \qquad\qquad \text{trace}(A^2 \circ P_k) = \binom{n}{k+1}(n H_1 H_{k+1} - (n-k-1)H_{k+2}),$$

where

$$c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1},$$

and $H_k = 0$ if $k > n$. We refer the reader to [2] for further details (see also [31] and [32] for other details about classical Newton tranformations for hypersurfaces in Riemannian spaces).

Let ∇ stand for the Levi-Civita connection of Σ. Associated to each Newton transformation P_k, we consider the second order linear differential operator $L_k : \mathcal{C}^\infty(\Sigma) \to \mathcal{C}^\infty(\Sigma)$ given by

$$L_k(f) = \text{trace}(P_k \circ \nabla^2 f).$$

Here $\nabla^2 f : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f, and it is given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathcal{X}(\Sigma).$$

Observe that

$$
\begin{aligned}
L_k(f) &= \text{trace}(P_k \circ \nabla^2 f) = \sum_{i=1}^{n} \langle P_k(\nabla_{E_i}\nabla f), E_i \rangle \\
&= \sum_{i=1}^{n} \langle \nabla_{E_i}\nabla f, P_k(E_i) \rangle = \sum_{i=1}^{n} \langle \nabla_{P_k(E_i)}\nabla f, E_i \rangle = \text{trace}(\nabla^2 f \circ P_k),
\end{aligned}
$$

where $\{E_1, \ldots, E_n\}$ is a local orthonormal frame on Σ. Moreover, we also have that

$$\operatorname{div}(P_k(\nabla f)) = \sum_{i=1}^{n} \langle (\nabla_{E_i} P_k)(\nabla f), E_i \rangle + \sum_{i=1}^{n} \langle P_k(\nabla_{E_i} \nabla f), E_i \rangle$$

(8)
$$= \langle \operatorname{div} P_k, \nabla f \rangle + L_k(f),$$

where

$$\operatorname{div} P_k := \operatorname{trace}(\nabla P_k) = \sum_{i=1}^{n} (\nabla_{E_i} P_k)(E_i).$$

In particular, if P_k is divergence-free then $L_k(f) = \operatorname{div}(P_k(\nabla f))$ and L_k is a divergence form differential operator on Σ. This happens trivially when $k = 0$, and actually L_0 is nothing but the Laplacian operator Δ. On the other hand, by [2, Corollary 3.2] this also happens for every $k = 0, \ldots, n$ when the GRW ambient spacetime has constant sectional curvature. This is a consequence of the following useful expression for the divergence of the Newton operators, which can be found in [2, Lemma 3.1].

Lemma 3.1. *The divergences of the Newton transformations P_k of a spacelike hypersurface Σ^n immersed into an arbitrary $(n+1)$-dimensional spacetime are given by*

$$\langle \operatorname{div} P_k, X \rangle = \sum_{j=0}^{k-1} \sum_{i=1}^{n} \langle \overline{R}(E_i, A^{k-1-j} X)N, P_j E_i \rangle, \quad k = 1, \ldots, n-1,$$

for every tangent field $X \in \mathcal{X}(\Sigma)$, where $\overline{R}$ stands for the curvature tensor of the ambient spacetime[1].

It follows from (8) that the operator L_k is elliptic if and only if P_k is positive definite. Clearly, $L_0 = \Delta$ is always elliptic. For our applications, it will be useful to have some geometric conditions which guarantee the ellipticity of L_k when $k \geq 1$. For $k = 1$ the following one is esentially a consequence of Lemma 3.10 in [20].

Lemma 3.2. *Let Σ be a spacelike hypersurface immersed into a GRW spacetime. If $H_2 > 0$ on Σ, then L_1 is elliptic or, equivalently, P_1 is positive definite (for an appropriate choice of the Gauss map N).*

Proof. Simple observe that by Cauchy-Schwarz inequality we have $H_1^2 \geq H_2 > 0$, and H_1 does not vanish on Σ. By choosing the appropriate Gauss map, we may assume that $H_1 > 0$. Recall that H_2 does not depend on the chosen N. Since $n^2 H_1^2 = \sum \kappa_i^2 + n(n-1)H_2 > \kappa_i^2$ for every $i = 1, \ldots, n$, then $\mu_{i,1} = nH_1 + \kappa_i > 0$ for every i and P_1 is positive definite. $\square$

On the other hand, by an elliptic point in a spacelike hypersurface we mean a point of Σ where all the principal curvatures are negative, with respect to an appropriate choice of the Gauss map N. When $2 \leq k \leq n - 1$, the existence of an elliptic point in Σ also implies here that the operator L_k is elliptic on Σ, under the assumption that H_{k+1} is positive on Σ for that choice of N (if k is even). Even more, we have the following result.

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[1] In our notation and following [30], we use $\overline{R}(X, Y)Z = \overline{\nabla}_{[X,Y]}Z - [\overline{\nabla}_X, \overline{\nabla}_Y]Z$

Lemma 3.3. *Let Σ be a spacelike hypersurface immersed into a GRW spacetime. If there exists an elliptic point of Σ, with respect to an appropriate choice of the Gauss map N, and $H_{k+1} > 0$ on Σ, for $2 \le k \le n-1$, then for all $1 \le j \le k$ the operator L_j is elliptic or, equivalently, P_j is positive definite (for that appropriate choice of the Gauss map, if j is odd).*

The proof follows from that of [14, Proposition 3.2] (see also the proof of [8, Proposition 3.2]), taking into account that in our case, and by our sign convention in the definition of the j-th mean curvatures, we have $\binom{n}{j}H_j = \sigma_j(-\kappa_1, \ldots, -\kappa_n) = (-1)^j S_j$.

4. The operator L_k acting on the height function

The vector field given by

$$K(t,x) = f(t)(\partial/\partial t)_{(t,x)}, \quad (t,x) \in -I \times_f M^n,$$

determines a non-vanishing future-pointing closed conformal vector field on $-I \times_f M^n$. In fact,

$$\overline{\nabla}_Z K = f'(t)Z \tag{9}$$

for every vector Z tangent to $-I \times_f M^n$ at a point (t,x), where $\overline{\nabla}$ denotes the Levi-Civita connection on $-I \times_f M^n$. This conformal field K will be an essential tool in our computations.

Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface with Gauss map N. The height function of Σ, denoted by h, is the restriction of the projection $\pi_I(t,x) = t$ to Σ, that is, $h : \Sigma \to I$ is given by $h = \pi_I \circ \psi$. Observe that the gradient of π_I on $-I \times_f M^n$ is given by

$$\overline{\nabla}\pi_I = -\langle \overline{\nabla}\pi_I, \partial_t \rangle \partial_t = -\partial_t.$$

Then, the gradient of h on Σ is given by

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top, \tag{10}$$

where

$$\partial_t = \partial_t^\top - \langle N, \partial_t \rangle N$$

and $\partial_t^\top \in \mathcal{X}(\Sigma)$ denotes the tangential component of ∂_t. In our computations, we will also consider the function $g(h)$, where $g : I \to \mathbb{R}$ is an arbitrary primitive of f. Since $g' = f > 0$, then $g(h)$ can be thought as a reparametrization of the height function. In particular, the gradient of $g(h)$ on Σ is

$$\nabla g(h) = f(h)\nabla h = -f(h)\partial_t = -K^\top, \tag{11}$$

where $K^\top$ denotes the tangential component of K along the hypersurface,

$$K = K^\top - \langle N, K \rangle N. \tag{12}$$

Equation (9) implies that

$$\overline{\nabla}_X K = f'(h)X \tag{13}$$

for every $X \in \mathcal{X}(\Sigma)$. From here, we can easily see that

$$\nabla_X K^\top = f'(h)X - f(h)\langle N, \partial_t \rangle AX = f'(h)X - \langle N, K \rangle AX. \tag{14}$$

Therefore, from (11) we get that

$$\nabla_X(\nabla g(h)) = -\nabla_X K^\top = -f'(h)X + \langle N, K \rangle AX, \tag{15}$$

and then, by (5) and (6), we conclude that

$$L_k(g(h)) = -f'(h)\mathrm{tr}(P_k) + \langle N, K \rangle \mathrm{tr}(A \circ P_k) = -c_k(f'(h)H_k + \langle N, K \rangle H_{k+1}).$$

On the other hand, taking into account that

$$\nabla h = \frac{1}{f(h)}\nabla g(h),$$

we also get from (15) that

$$(16) \qquad \nabla_X(\nabla h) = -(\log f)'(h)(X + \langle \nabla h, X \rangle \nabla h) + \langle N, \partial_t \rangle AX,$$

and therefore

$$
\begin{aligned}
L_k(h) &= -(\log f)'(h)(\mathrm{tr}(P_k) + \langle P_k(\nabla h), \nabla h \rangle) + \langle N, \partial_t \rangle \mathrm{tr}(A \circ P_k) \\
&= -(\log f)'(h)(c_k H_k + \langle P_k(\nabla h), \nabla h \rangle) - \langle N, \partial_t \rangle c_k H_{k+1}.
\end{aligned}
$$

For future reference, we summarize this in the following lemma.

Lemma 4.1. *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N. Let $h = \pi_I \circ \psi$ denote the height function of Σ, and let $g : I \to \mathbb{R}$ be any primitive of f. Then, for every $k = 0, \ldots, n-1$ we have*

$$(17) \qquad L_k(h) = -(\log f)'(h)(c_k H_k + \langle P_k(\nabla h), \nabla h \rangle) - \langle N, \partial_t \rangle c_k H_{k+1},$$

and

$$(18) \qquad L_k(g(h)) = -c_k(f'(h)H_k + \langle N, K \rangle H_{k+1}),$$

where

$$c_k = (n-k)\binom{n}{k} = (k+1)\binom{n}{k+1}.$$

5. First applications

As a first application of Lemma 4.1 we will prove Theorem 5.1 below, which extends Theorem 7 in [28] (see also Theorem 1 in [3]) for hypersurfaces with constant mean curvature to the case of hypersurfaces for which the quotient H_{k+1}/H_k is constant, where $k = 0, 1, \ldots, n-1$. Before stating our result, recall from [4, Proposition 3.2 (i)] that if a GRW spacetime admits a compact spacelike hypersurface, then the Riemannian factor M^n is necessarily compact. In that case, it is said that $-I \times_f M^n$ is a spatially closed GRW spacetime.

Theorem 5.1. *Let $-I \times_f M^n$ a spatially closed GRW spacetime such that its warping function satisfies the following condition*

$$(19) \qquad\qquad\qquad ff'' - f'^2 \leq 0,$$

(that is, such that $-\log f$ is convex). Let Σ^n be a compact spacelike hypersurface immersed into $-I \times_f M^n$ whose k-th Newton transformation P_k is definite on Σ, for some $k = 0, 1, \ldots, n-1$. If the quotient H_{k+1}/H_k is constant, then the hypersurface is an embedded slice $\{t_0\} \times M$, where $t_0 \in I$ satisfies $f'(t_0) \neq 0$ if $k \geq 1$.

As observed by Montiel in [28], condition (19) is a reasonably weak condition on the warping function which is sufficient in order to obtain uniqueness results for spacelike hypersurfaces in GRW spacetimes. For instance, every GRW spacetime obeying the timelike convergence condition satisfies (19).

Before giving the proof of Theorem 5.1, it is interesting to obtain some applications of it for situations where the definiteness of P_k can be derived from geometric hypothesis. For instance, from Lemma 3.2 we have the following.

Corollary 5.2. *Let $-I \times_f M^n$ a spatially closed GRW spacetime such that its warping function satisfies the condition*

$$f f'' - f'^2 \leq 0.$$

Then the only compact spacelike hypersurfaces Σ immersed into $-I \times_f M^n$ such that $H_2 > 0$ on Σ and the quotient H_2/H_1 is constant are the embedded slices $\{t_0\} \times M$, with $t_0 \in I$ satisfying $f'(t_0) \neq 0$.

On the other hand, in order to apply Lemma 3.3, it is convenient to have some geometric condition which implies the existence of an elliptic point. The following technical result is a consequence of a more general result given by the authors in [2, Lemma 5.4 and Remark 5.7], jointly with Brasil Jr., and it will be essential in most of our applications. For that reason and for the sake of completeness we give here a proof of it, especially adapted to the case of hypersurfaces in GRW spacetimes.

Lemma 5.3 (Existence of an elliptic point)**.** *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a compact spacelike hypersurface immersed into a spatially closed GRW spacetime, and assume that $f'(h)$ does not vanish on Σ (equivalently, $\psi(\Sigma)$ is contained in a slab*

$$\Omega(t_1, t_2) = (t_1, t_2) \times M \subset -I \times_f M^n$$

on which f' does not vanish)
  (1) *If $f'(h) > 0$ on Σ (equivalently, $f' > 0$ on (t_1, t_2)), then there exists an elliptic point of Σ with respect to its future-pointing Gauss map.*
  (2) *If $f'(h) < 0$ on Σ (equivalently, $f' < 0$ on (t_1, t_2)), then there exists an elliptic point of Σ with respect to its past-pointing Gauss map.*

Proof. When $f'(h) > 0$, let us choose on Σ the future-pointing Gauss map N and let $p_0 \in \Sigma$ be the point where the height function h attains its minimum on Σ. Then we have $\nabla h(p_0) = 0$, $\langle N, \partial_t \rangle(p_0) = -1$ and by (16) we get

$$\nabla^2 h_{p_0}(e_i, e_i) = -(\log f)'(h_{\min}) - \kappa_i(p_0) \geq 0$$

for every $i = 1, \ldots, n$, where $\{e_1, \ldots, e_n\}$ is a basis of principal directions at p_0. Since $f'(h_{\min}) > 0$ we get from here

$$\kappa_i(p_0) \leq -(\log f)'(h_{\min}) < 0,$$

as dessired. On the other hand, when $f'(h) < 0$ we choose on Σ the past-pointing Gauss map N and consider $p_0 \in \Sigma$ the point where the height function h attains its maximum on Σ. Now we have $\nabla h(p_0) = 0$, $\langle N, \partial_t \rangle(p_0) = 1$ and by (16) we get

$$\nabla^2 h_{p_0}(e_i, e_i) = -(\log f)'(h_{\max}) + \kappa_i(p_0) \leq 0$$

for every $i = 1, \ldots, n$, with $\{e_1, \ldots, e_n\}$ a basis of principal directions at p_0. Since now $f'(h_{\max}) < 0$ we get from here

$$\kappa_i(p_0) \leq (\log f)'(h_{\min}) < 0.$$

$\square$

Now, using Lemma 3.3 and Lemma 5.3 we can also state the following application of Theorem 5.1.

Corollary 5.4. *Let $-I \times_f M^n$ a spatially closed GRW spacetime such that its warping function satisfies the condition*

$$ff'' - f'^2 \leq 0.$$

Assume that Σ^n, $n \geq 3$, is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which is contained in a slab $\Omega(t_1, t_2) \subset -I \times_f M^n$ on which f' does not vanish. If $H_{k+1} > 0$ on Σ for some $k \geq 2$ and one of the the quotients H_{j+1}/H_j is constant for some $1 \leq j \leq k$, then Σ is necessarily an embedded slice $\{t_0\} \times M$, with $t_0 \in (t_1, t_2)$.

The proof of our Theorem 5.1 will be a consequence of the following auxiliary result, which is a generalization of Lemma 3 and Corollary 4 in [3].

Lemma 5.5. *Let $-I \times_f M^n$ a spatially closed GRW spacetime, and let Σ^n be a compact spacelike hypersurface immersed into $-I \times_f M^n$. Assume that for some $k \in \{0, 1 \ldots, n-1\}$, the k-th Newton transformation P_k is semi-definite on Σ and the function H_k does not vanish on Σ. Then the quotient H_{k+1}/H_k satifies*

$$(20) \qquad \min\left(\frac{H_{k+1}}{H_k}\right) \leq (\log f)'(h_{\max}) \quad \text{and} \quad \max\left(\frac{H_{k+1}}{H_k}\right) \geq (\log f)'(h_{\min}),$$

where $h_{\min}$ and $h_{\max}$ denote, respectively, the minimum and the maximum values of the height function on Σ. In particular, if the warping function satifies condition (19) then

$$(21) \qquad \min\left(\frac{H_{k+1}}{H_k}\right) \leq (\log f)'(h_{\max}) \leq (\log f)'(h_{\min}) \leq \max\left(\frac{H_{k+1}}{H_k}\right).$$

Proof. Choose on Σ its future-pointing Gauss map N. Since Σ is compact, there exist points $p_{\min}, p_{\max} \in \Sigma$ where the height function attains its minimum and its maximum values, respectively, that is,

$$h(p_{\min}) = \min_{\Sigma} h = h_{\min} \leq h_{\max} = h(p_{\max}) = \max_{\Sigma} h.$$

In particular, $\nabla h(p_{\min}) = 0$ and $\nabla h(p_{\max}) = 0$, and from (10) we have $N(p_{\min}) = (\partial_t)_{p_{\min}}$ and $N(p_{\max}) = (\partial_t)_{p_{\max}}$. Using this in (17), we obtain that

$$(22) \qquad (1/c_k)L_k(h)(p_{\min}) = H_{k+1}(p_{\min}) - (\log f)'(h_{\min})H_k(p_{\min}),$$

and

$$(23) \qquad (1/c_k)L_k(h)(p_{\max}) = H_{k+1}(p_{\max}) - (\log f)'(h_{\max})H_k(p_{\max}).$$

Let us consider first the case where P_k is positive semi-definite on Σ. Then $H_k > 0$ on Σ, because we are also assuming that $H_k \neq 0$ on Σ. In that case, taking inot account that $L_k(h) = \mathrm{tr}(P_k \circ \nabla^2 h)$, $\nabla^2 h(p_{\min})$ is positive semi-definite and $\nabla^2 h(p_{\max})$ is negative semi-definite, then we have that $L_k(h)(p_{\min}) \geq 0$ and $L_k(h)(p_{\max}) \leq 0$. Therefore, equations (22) and (23) imply

$$(24) \qquad \max\left(\frac{H_{k+1}}{H_k}\right) \geq \frac{H_{k+1}}{H_k}(p_{\min}) \geq (\log f)'(h_{\min}),$$

and

$$(25) \qquad \min\left(\frac{H_{k+1}}{H_k}\right) \leq \frac{H_{k+1}}{H_k}(p_{\max}) \leq (\log f)'(h_{\max}).$$

This yields (20). Finally, if $-\log f$ is convex, then $(\log f)'$ is non increasing and so

$$(\log f)'(h_{\max}) \leq (\log f)'(h_{\min}),$$

which jointly with (20) yields (21). This completes the proof of Lemma 5.5 in the case where P_k is positive semi-definite. When P_k is negative semi-definite, then the proof above also works, taking into account that now $H_k < 0$ on Σ, and $L_k(h)(p_{\min}) \leq 0$ and $L_k(h)(p_{\max}) \geq 0$. $\qquad\square$

Proof of Theorem 5.1. Let us choose again the future-pointing Gauss map N on Σ. Since the quotient H_{k+1}/H_k is constant on Σ, we know from (21) that $H_{k+1}/H_k = (\log f)'(h_{\max}) = (\log f)'(h_{\min})$. Therefore

$$(\log f)'(h) = \frac{f'(h)}{f(h)} = H_{k+1}/H_k = \alpha$$

is constant on Σ, because $(\log f)'$ is non increasing on I by the hypothesis on the convexity of $-\log f$. Then, (18) reduces to

$$(26) \qquad\qquad L_k(g(h)) = -c_k(f(h) + \langle N, K \rangle)H_{k+1}.$$

Observe that $H_{k+1} = \alpha H_k$ does not change sign on Σ, and observe also that the function $\langle N, K \rangle$ satisfies

$$(27) \qquad\qquad \langle N, K \rangle \leq -\|K\| = -f(h) < 0 \quad \text{on} \quad \Sigma.$$

Hence, by (26) we have that $L_k(g(h))$ does not change sign on Σ, which is a compact Riemannian manifold. In the case where P_k is positive definite, then L_k is an elliptic operator, and we may apply the classical maximum principle for such operators to conclude that the function $g(h)$ is constant on the compact Σ. If P_k is negative definite, then the operator $-L_k$ is now elliptic and by the same argument we also conclude that $g(h)$ is constant. But $g(t)$ being an increasing function on t, that means that the height function h is constant on Σ and then the hypersurface is an slice $\{t_0\} \times M$. Finally, when $k > 0$ it necessarily holds $f'(t_0) \neq 0$ because slices $\{t\} \times M$ with $f'(t) = 0$ are totally geodesic in $-I \times_f M^n$, and then $P_k = 0$ for every $k > 0$. $\qquad\square$

It is worth pointing out that if the warping function satisfies the stronger condition $ff'' - f'^2 \leq 0$ with equality only at isolated points of I at most, then Theorem 5.1 remains true under the assumption that P_k is semi-definite and the function H_k does not vanish on Σ. Actually, in this case we also get from (21) in Lemma 5.5 that $H_{k+1}/H_k = (\log f)'(h_{\max}) = (\log f)'(h_{\min})$. But the hypothesis on the warping function implies now that $(\log f)'$ is strictly decreasing on I. Therefore, $h_{\min} = h_{\max}$ and the height function h is constant on Σ. Then, we also obtain the following related result.

Theorem 5.6. *Let $-I \times_f M^n$ a spatially closed GRW spacetime such that its warping function satisfies the condition (19), with equality only at isolated points of I at most. Let Σ^n be a compact spacelike hypersurface immersed into $-I \times_f M^n$ whose k-th Newton transformation P_k is semi-definite on Σ and H_k does not vanish on Σ, for some $k = 0, 1 \ldots, n-1$. If the quotient H_{k+1}/H_k is constant, then the hypersurface is an embedded slice $\{t_0\} \times M$, where t_0 satisfies $f'(t_0) \neq 0$ if $k > 0$.*

6. Minkowski formulae for hypersurfaces in GRW spacetimes

The use of Minkowski-type integral formulae for compact spacelike hypersurfaces in Lorentzian ambient spacetimes was first started by Montiel in [27] in his study of hypersurfaces with constant mean curvature in de Sitter space, and it was continued later by Alías, Romero and Sánchez in [4, 5, 6] for constant mean curvature hypersurfaces in GRW spacetimes and, more generally, in conformally stationary spacetimes. Observe that for the case of the mean curvature, only the first and second Minkowski formulae come into play. In [1], Aledo, Alías and Romero developed the general Minkowski formulae for compact spacelike hypersurfaces in de Sitter space, and applied them to the study of hypersurfaces with constant k-th mean curvature. On the other hand, in [28] Montiel gave also another proof of the first and second Minkowski formulae for spacelike hypersurfaces in conformally stationary spacetimes with a closed conformal field, as well as a proof of the third Minkowski formula for the case where the ambient spacetime has constant sectional curvature. As observed by Montiel, his method of proof, which follows the ideas of Hsiung [25] and uses parallel hypersurfaces, has a very nice geometric interpretation but it is very difficult to carry our for succesive Minkowski formulae. The reason is that it involves covariant derivatives of the Ricci tensor of the ambient space, and one need to assume at least that the ambient space is Einstein to get some control of them. In [2] the authors, jointly with Brasil Jr., developed another method to obtain general Minkowski formulae for spacelike hypersurfaces in conformally stationary spacetimes. Although more analytic than geometric, this method, which follows the ideas of Reilly in [31], has the advantage of working for succesive Minkowski formulae. However, in order to get some interesting applications from them, one need to assume again that the ambien t space has constant sectional curvature.

In this section, and as another application of our formulae in Lemma 4.1, we will derive general Minkowski-type formulae for compact spacelike hypersurfaces immersed into a GRW spacetime. The main interest of these new approach is that our new Minkowski formulae can be applied to the study of hypersurfaces with constant higher order mean curvature in arbitrary Robertson-Walker spacetimes, even when the ambient space does not have constant sectional curvature.

Let Σ^n be a compact spacelike hypersurface immersed into a GRW spacetime. The general Minkowski formulae for Σ, as derived in [2], are just a simple consequence of our formula (18) and the expression (8). In fact, it follows from (8) and (18) that

$$\operatorname{div}(P_k(\nabla g(h))) = f(h)\langle \operatorname{div} P_k, \nabla h \rangle - c_k(f'(h)H_k + \langle N, K \rangle H_{k+1}),$$

for every $k = 0, \ldots, n - 1$. Therefore, integrating this equality on the compact Riemannian manifold Σ, the divergence theorem gives the following Minkowski formulae,

$$(28) \qquad c_k \int_\Sigma (f'(h)H_k + \langle N, K \rangle H_{k+1})d\Sigma = \int_\Sigma f(h)\langle \operatorname{div} P_k, \nabla h \rangle d\Sigma,$$

where $d\Sigma$ is the Riemannian measure induced on Σ. This is just Theorem 4.1 in [2] for the case of hypersurfaces into a GRW spacetime. In particular, if $k = 0$ then

$P_0 = I$, div$P_0 = 0$, and this gives the first Minkowski formula

$$\int_\Sigma (f'(h) + \langle N, K\rangle H_1)d\Sigma = 0,$$

which holds true without any additional hypothesis on the ambient GRW spacetime. On the other hand, when $k = 1$ we easily get from the inductive formula in Lemma 3.1 that

$$\langle \text{div}P_1, \nabla h\rangle = \sum_{i=1}^n \langle \overline{R}(E_i, \nabla h)N, E_i\rangle = -\overline{\text{Ric}}(N, \nabla h),$$

where $\overline{\text{Ric}}$ denotes the Ricci tensor of $-I \times_f M^n$. A direct computation using the general relationship between the Ricci tensor of a warped product and the Ricci tensors of its base and its fiber, as well as the derivatives of the warping function (see for instance [30, Corollary 43]), implies that, for the special case of our GRW ambient spacetimes, we have

$$\overline{\text{Ric}}(U, V) \quad = \quad \text{Ric}_M(U^*, V^*) + (n((\log f)')^2 + (\log f)'')\langle U, V\rangle$$

(29)
$$-(n-1)(\log f)''\langle U, \partial_t\rangle\langle V, \partial_t\rangle,$$

for arbitrary vector fields U, V on $-I \times_f M^n$, where, for simplicity, we are writing $\log f = \log f(\pi_I)$, $(\log f)' = (\log f)'(\pi_I)$, and $(\log f)'' = (\log f)''(\pi_I)$. Here Ric_M denotes the Ricci tensor of the Riemannian manifold M^n, and $U^* = (\pi_M)_*(U)$ denotes projection onto the fiber M of a vector field U defined on $-I \times_f M^n$, that is,

$$U = U^* - \langle U, \partial_t\rangle\partial_t.$$

Observe that the decomposition $\partial_t = -\nabla h - \langle N, \partial_t\rangle N$ yields $(\nabla h)^* = -\langle N, \partial_t\rangle N^*$. In particular, it follows from (29) that

$$\langle \text{div}P_1, \nabla h\rangle = -\overline{\text{Ric}}(N, \nabla h) = \langle N, \partial_t\rangle\left(\text{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2\right).$$

Therefore, the second Minkowski formula (when $k = 1$) for a compact spacelike hypersurface into a GRW spacetime becomes

$$c_1 \int_\Sigma (f'(h)H_1 + \langle N, K\rangle H_2)d\Sigma =$$

$$\int_\Sigma \langle N, K\rangle \left(\text{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2\right) d\Sigma.$$

For its further reference, we summarize here the first two Minkowski formulae for spacelike hypersurfaces in a GRW spacetime.

Theorem 6.1. *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a compact spacelike hypersurface immersed into a GRW spacetime. Then*

(30)
$$\int_\Sigma (f'(h) + \langle N, K\rangle H_1)d\Sigma = 0,$$

and

(31)
$$n(n-1) \int_\Sigma (f'(h)H_1 + \langle N, K\rangle H_2)d\Sigma =$$

$$\int_\Sigma \langle N, K\rangle \left(\text{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2\right) d\Sigma.$$

In order to find a useful expression for the succesive Minkowski formulae given by (28), in the case where $n \geq 3$, we will assume that $-I \times_f M^n$ is a (classical) Robertson-Walker spacetime, that is, the Riemannian manifold M^n has constant sectional curvature κ. Our objective is to compute $\langle \mathrm{div} P_k, \nabla h \rangle$ when $2 \leq k \leq n-1$. From Lemma 3.1 we know that

$$(32) \qquad \langle \mathrm{div} P_k, \nabla h \rangle = \sum_{j=0}^{k-1} \sum_{i=1}^{n} \langle \overline{R}(E_i, A^{k-1-j}(\nabla h))N, P_j E_i \rangle,$$

where $\{E_1, \ldots, E_n\}$ is an arbitrary local orthonormal frame on Σ. A direct but heavy computation using the general relationship between the curvature tensor of a warped product and the curvature tensors of its base and its fiber, as well as the derivatives of the warping function (see for instance [30, Proposition 42]), implies that, for our GRW ambient spacetimes, we have the following expression

$$
\begin{aligned}
(33) \quad \overline{R}(U,V)W = {} & R_M(U^*, V^*)W^* + ((\log f)')^2 (\langle U, W \rangle V - \langle V, W \rangle U) \\
& + (\log f)'' \langle W, \partial_t \rangle (\langle V, \partial_t \rangle U - \langle U, \partial_t \rangle V) \\
& - (\log f)'' (\langle V, \partial_t \rangle \langle U, W \rangle - \langle U, \partial_t \rangle \langle V, W \rangle) \partial_t,
\end{aligned}
$$

for arbitrary vector fields U, V, W on $-I \times_f M^n$. Here R_M denotes the curvature tensor of the Riemannian manifold M^n, which in principle we do not assume to be of constant sectional curvature. In particular, (33) implies

$$(34) \quad \overline{R}(E_i, X)N = R_M(E_i^*, X^*)N^* - (\log f)''(h)\langle N, \partial_t \rangle (\langle X, \nabla h \rangle E_i - \langle E_i, \nabla h \rangle X)$$

for every tangent vector field $X \in \mathcal{X}(\Sigma)$. From the decompositions

$$N = N^* - \langle N, \partial_t \rangle \partial_t, \quad E_i = E_i^* - \langle E_i, \partial_t \rangle \partial_t, \quad \text{and} \quad X = X^* - \langle X, \partial_t \rangle \partial_t,$$

it follows easily that

$$(35) \qquad (N^*)^\top = -\langle N, \partial_t \rangle \nabla h,$$

$$(36) \qquad (E_i^*)^\top = E_i + \langle E_i, \nabla h \rangle \nabla h,$$

$$(37) \qquad (X^*)^\top = X + \langle X, \nabla h \rangle \nabla h,$$

and

$$(38) \qquad \langle N^*, N^* \rangle_M = \frac{1}{f^2(h)} \|\nabla h\|^2,$$

$$(39) \qquad \langle N^*, E_i^* \rangle_M = -\frac{1}{f^2(h)} \langle N, \partial_t \rangle \langle E_i, \nabla h \rangle,$$

$$(40) \qquad \langle N^*, X^* \rangle_M = -\frac{1}{f^2(h)} \langle N, \partial_t \rangle \langle X, \nabla h \rangle.$$

Let us remark here, for its use later, that this set of equalities ((35)-(40)) holds true for every Riemannian fiber M^n, since we are not using yet that M^n has constant sectional curvature.

If M^n has constant sectional curvature κ, then we also have that

$$(R_M(E_i^*, X^*)N^*)^\top = \kappa \langle E_i^*, N^* \rangle_M (X^*)^\top - \kappa \langle X^*, N^* \rangle_M (E_i^*)^\top.$$

Then, using equations (36) and (37) we obtain

$$(R_M(E_i^*, X^*)N^*)^\top = \frac{\kappa}{f^2(h)} \langle N, \partial_t \rangle \left(\langle X, \nabla h \rangle E_i - \langle E_i, \nabla h \rangle X \right),$$

which jointly with (34) yields

$$(\overline{R}(E_i, X)N)^\top = \left(\frac{\kappa}{f^2(h)} - (\log f)''(h)\right) \langle N, \partial_t\rangle \left(\langle X, \nabla h\rangle E_i - \langle E_i, \nabla h\rangle X\right).$$

Therefore, for every fixed $j = 0, \ldots, k-1$ we find

$$\sum_{i=1}^{n} \langle \overline{R}(E_i, X)N, P_j E_i\rangle =$$

$$\left(\frac{\kappa}{f^2(h)} - (\log f)''(h)\right) \langle N, \partial_t\rangle \left(\operatorname{tr}(P_j)\langle X, \nabla h\rangle - \langle P_j X, \nabla h\rangle\right)$$

for every tangent vector field $X \in \mathcal{X}(\Sigma)$. Using this expression into (32), we get

$$\langle \operatorname{div} P_k, \nabla h\rangle = \left(\frac{\kappa}{f^2(h)} - (\log f)''(h)\right) \langle N, \partial_t\rangle$$

(41)
$$\sum_{j=0}^{k-1} \left(\operatorname{tr}(P_j)\langle A^{k-1-j}\nabla h, \nabla h\rangle - \langle (P_j \circ A^{k-1-j})\nabla h, \nabla h\rangle\right).$$

Now we claim that, in general,

(42)
$$\sum_{j=0}^{k-1} \left(\operatorname{tr}(P_j)A^{k-1-j} - P_j \circ A^{k-1-j}\right) = (n-k)P_{k-1},$$

for every $2 \leq k \leq n$. We will prove (42) by induction on k. When $k = 2$, using (5) we easily see that (42) reduces to

$$(n-1)A + n(n-1)H_1 I - P_1 = (n-1)P_1 - P_1 = (n-2)P_1.$$

Now assume that (42) holds true for $k - 1 \geq 2$, and write

$$\sum_{j=0}^{k-1} \left(\operatorname{tr}(P_j)A^{k-1-j} - P_j \circ A^{k-1-j}\right) = \sum_{j=0}^{k-2} \left(\operatorname{tr}(P_j)A^{k-2-j} - P_j \circ A^{k-2-j}\right) \circ A$$
$$+ \operatorname{tr}(P_{k-1})I - P_{k-1}.$$

Using the induction hypothesis here we get that

$$\sum_{j=0}^{k-1} \left(\operatorname{tr}(P_j)A^{k-1-j} - P_j \circ A^{k-1-j}\right) = (n-k+1)P_{k-2} \circ A$$
$$+ (n-k+1)\binom{n}{k-1}H_{k-1}I - P_{k-1}$$
$$= (n-k+1)P_{k-1} - P_{k-1}$$
$$= (n-k)P_{k-1},$$

as dessired. Finally, using (42) into (41) we conclude that

(43) $$\langle \operatorname{div} P_k, \nabla h\rangle = (n-k)\left(\frac{\kappa}{f^2(h)} - (\log f)''(h)\right) \langle N, \partial_t\rangle \langle P_{k-1}\nabla h, \nabla h\rangle$$

for every $k \geq 2$. Putting this expression into (28), we obtain the following version of higher order Minkowski formulae for spacelike hypersurfaces in RW spacetimes.

Theorem 6.2. *Let $\psi : \Sigma^n \to -I \times_f M^n$ be a compact spacelike hypersurface immersed into a RW spacetime, M^n being a Riemannian manifold with constant sectional curvature κ. Then, for every $k = 2, \ldots, n-1$*

$$\binom{n}{k} \int_\Sigma (f'(h)H_k + \langle N, K \rangle H_{k+1})d\Sigma =$$

(44)
$$\int_\Sigma \left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle N, K \rangle \langle P_{k-1}\nabla h, \nabla h \rangle d\Sigma.$$

Formulas (30) and (31) are refered to as the first and second Minkowski formulae, respectively. In the following, we will refer to formula (44) as the $(k+1)$-th Minkowski formula, with $k = 2, \ldots, n-1$.

7. Umbilicity of hypersurfaces in RW spacetimes

In [28, Theorem 8] Montiel gave a uniqueness result for compact spacelike hypersurfaces with constant scalar curvature immersed into a constant sectional curvature spacetime which is equipped with a closed conformal timelike vector field (see also [2, Theorem 5.3] for the case where the conformal timelike vector field is not necessarily closed). As observed by Montiel in [28], every spacetime having a closed conformal timelike vector field is locally isometric to a GRW spacetime. On the other hand, it is not difficult to see that a GRW spacetime $-I \times_f M^n$ has constant sectional curvature $\bar{\kappa}$ if and only if the Riemannian M^n has constant sectional curvature κ (that is, $-I \times_f M^n$ is in fact a RW spacetime) and the warping function f satisfies the following differential equations

$$\frac{f''}{f} = \bar{\kappa} = \frac{(f')^2 + \kappa}{f^2}$$

(see, for instance, [7, Corollary 9.107]). Therefore, Montiel's result [28, Theorem 8] esentially states as follows.

Theorem 7.1. *Let $-I \times_f M^n$ be a spatially closed RW spacetime with constant sectional curvature $\bar{\kappa}$, with $n \geq 3$. Then every compact spacelike hypersurface Σ immersed into $-I \times_f M^n$ with constant scalar curvature S such that $S < n(n-1)\bar{\kappa}$ is totally umbilical. Moreover, if $\bar{\kappa} \leq 0$ then Σ is an embedded slice $\{t\} \times M^n$ (necessarily with $f'(t) \neq 0$), and if $\bar{\kappa} > 0$ then Σ is either an embedded slice $\{t\} \times M^n$ (necessarily with $f'(t) \neq 0$) or a round umbilical hypersphere in a de Sitter space.*

As a first application of our Minkowski formulae in Theorem 6.2, we extend this result to the case of hypersurfaces immersed into a RW spacetime obeying the null convergence condition (NCC). Recall that a spacetime obeys NCC if its Ricci curvature is non-negative on null (or lightlike) directions. Obviously, every spacetime with constant sectional curvature trivially obeys NCC. Using expression (29), it is not difficult to see that a GRW spacetime $-I \times_f M^n$ obeys NCC if and only if

(45)
$$\text{Ric}_M \geq (n-1) \sup_I (ff'' - f'^2)\langle,\rangle_M,$$

where Ric_M and $\langle,\rangle_M$ are respectively the Ricci and metric tensors of the Riemannian manifold M^n. In particular, a RW spacetime obeys NCC if and only

if

(46) $$\kappa \geq \sup_{I}(ff'' - f'^2),$$

where κ denotes the constant sectional curvature of M^n. On the other hand, the condition that the scalar curvature S of Σ is constant and less that $n(n-1)\bar{\kappa}$ in Theorem 7.1 is equivalent, from equation (4), to the fact that H_2 is a positive constant. Then, Theorem 7.1 admits the following extension.

Theorem 7.2. *Let $-I \times_f M^n$ be a spatially closed RW spacetime obeying the null convergence condition, with $n \geq 3$. Then every compact spacelike hypersurface immersed into $-I \times_f M^n$ with positive constant H_2 is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if the inequality in (46) is strict.*

Proof. Multiplying the first Minkowski formula (30) by the constant $\binom{n}{2}H_2$, and substracting the third Minkowski formula (formula (44) with $k = 2$), we obtain
(47)
$$\int_\Sigma \left(\binom{n}{2}(H_1 H_2 - H_3) + \left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle P_1 \nabla h, \nabla h \rangle \right) \langle N, K \rangle d\Sigma = 0.$$

Since $H_2 > 0$ we know from Lemma 3.2 that, for an appropriate choice of the Gauss map N, the transformation P_1 is positive definite and $H_1 > 0$ on Σ. Since we always have $H_2^2 - H_1 H_3 \geq 0$ (see, for instance, [24, Theorem 51 and Theorem 144]), then in our case we get

$$H_1 H_2 - H_3 \geq \frac{H_2}{H_1}(H_1^2 - H_2) \geq 0,$$

by Cauchy-Schwarz inequality, with equality if and only if Σ is totally umbilical. On the other hand, since P_1 is positive definite, we also have by NCC (46) that

$$\left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle P_1 \nabla h, \nabla h \rangle \geq 0.$$

Therefore, since $\langle N, K \rangle$ does not vanish on Σ, we conclude from (47) that Σ is totally umbilical and that

$$\left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle P_1 \nabla h, \nabla h \rangle \equiv 0 \quad \text{on} \quad \Sigma.$$

Since P_1 is positive definite, it follows from here that either Σ is a slice (equivalently, $\nabla h \equiv 0$) or

(48) $$\frac{\kappa}{f^2(h)} - (\log f)''(h) \equiv 0 \quad \text{on} \quad \Sigma.$$

First of all, observe that if Σ is a slice $\{t_0\} \times M^n$, then necessarily $f'(t_0) \neq 0$ because of the condition $H_2 > 0$ (recall that slices with $f'(t_0) = 0$ are totally geodesic). On the other hand, if Σ is not a slice, since it we already know that it is totally umbilical we have that $H_2 = H_1^2$, which implies that Σ has also constant mean curvature. Therefore, applying the classification of compact umbilical spacelike hypersurfaces with constant mean curvature given by Montiel in [28, Theorem 5], we conclude that the only case in which Σ is not a slice occurs only when $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere.

Finally, observe that if the inequality in (46) is strict, then (48) cannot occur and Σ is necessarily a slice. $\qquad\square$

For the general case of hypersurfaces with constant higher order mean curvature, we can state the following version of Theorem 7.2.

Theorem 7.3. *Let $-I \times_f M^n$ be a spatially closed RW spacetime obeying the null convergence condition, with $n \geq 3$. Assume that Σ^n is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which is contained in a slab $\Omega(t_1, t_2) \subset -I \times_f M^n$ on which f' does not vanish. If H_k is constant, with $3 \leq k \leq n$ then Σ is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if the inequality in (46) is strict.*

Proof. The proof follows the ideas of that of Theorem 5.10 in [2]. Let us assume, for instance, that $f' > 0$ on (t_1, t_2). From Lemma 5.3, we know that, with respect to the future-pointing Gauss map N, there exists a point $p_0 \in \Sigma$ where all the principal curvatures are negative. Therefore, the constant $H_k = H_k(p_0)$ is positive, and following the ideas of Montiel and Ros in [29, Lemma 1] and their use of Garding inequalities [21] (taking into account our sign convention in the definition of the j-th mean curvatures) we get that

$$(49) \qquad H_1 \geq H_2^{1/2} \geq \cdots \geq H_{k-1}^{1/(k-1)} \geq H_k^{1/k} > 0 \quad \text{on} \quad \Sigma.$$

Moreover, equality at any stage happens only at umbilical points. Therefore, since $f'(h) > 0$ on Σ, we have that $f'(h)H_{k-1} \geq f'(h)H_k^{(k-1)/k}$. Integrating this inequality, and using the first (30) and the k-th Minkowski formulae (that is, (44) with $k-1$ instead of k; observe that $2 \leq k-1 \leq n-1$), we get

$$H_k \int_\Sigma \langle N, K \rangle d\Sigma = -\int_\Sigma f'(h)H_{k-1}d\Sigma \leq$$

$$-H_k^{(k-1)/k} \int_\Sigma f'(h)d\Sigma = H_k^{(k-1)/k} \int_\Sigma \langle N, K \rangle H_1 d\Sigma.$$

That is,

$$(50) \qquad \int_\Sigma (H_1 - H_k^{1/k})\langle N, K \rangle d\Sigma \geq 0.$$

Since N is future-pointing, we know that $\langle N, K \rangle \leq -f(h) < 0$, and by (49) we have $H_1 - H_k^{1/k} \geq 0$. Therefore, (50) implies that $H_1 - H_k^{1/k} \equiv 0$ on Σ, and hence the hypersurface is totally umbilical. Since H_k is a positive constant and Σ is totally umbilical, we know that all the higher order mean curvatures are constant. In particular, H_2 is a positive constant and the result follows by applying Theorem 7.2. $\qquad\square$

When $n \geq 4$ and H_k is constant with $k \leq n-1$, there is still another version of Theorem 7.3 under the assumption of existence of an elliptic point.

Theorem 7.4. *Let $-I \times_f M^n$ be a spatially closed RW spacetime obeying the null convergence condition, with $n \geq 4$. Assume that Σ^n is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which contains an elliptic point. If H_k is constant, with $3 \leq k \leq n-1$ then Σ is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$*

has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if the inequality in (46) is strict.

Proof. The proof follows the ideas of that of Theorem 7.2. Multiplying now the first Minkowski formula (30) by the constant $\binom{n}{k}H_k$, and substracting the $(k+1)$-th Minkowski formula (formula (44); recall that $3 \le k \le n-1$)), we obtain

(51)
$$\int_\Sigma \left(\binom{n}{k}(H_1 H_k - H_{k+1}) + \left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle P_{k-1}\nabla h, \nabla h \rangle \right) \langle N, K \rangle d\Sigma = 0.$$

Assume that $p_0 \in \Sigma$ is an elliptic point, that is, a point where all the principal curvatures are negative. Therefore, the constant $H_k = H_k(p_0)$ is positive, and following the ideas of Montiel and Ros in [29, Lemma 1] and their use of Garding inequalities [21], as in the proof of Theorem 7.3, we get (49), with equality at any stage only at umbilical points. On the other hand, for every $1 \le j \le n$ one has the following generalized Cauchy-Schwarz inequalities (see, for instance, [24, Theorem 51 and Theorem 144]),

$$H_{j-1}H_{j+1} \le H_j^2.$$

Since each $H_j > 0$ for $j = 1, \ldots, k$ by (49), this is equivalent to

$$\frac{H_{k+1}}{H_k} \le \frac{H_k}{H_{k-1}} \le \cdots \frac{H_2}{H_1} \le H_1.$$

But this implies

(52)
$$H_1 H_k - H_{k+1} \ge 0,$$

with equality if and only if Σ is totally umbilical.

On the other hand, by Lemma 3.3 we also know that the transformation P_{k-1} is positive definite on Σ, which jointly with (46) yields

$$\left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle P_{k-1}\nabla h, \nabla h \rangle \ge 0.$$

Therefore, since $\langle N, K \rangle$ does not vanish on Σ, we conclude from (51) that Σ is totally umbilical and that

$$\left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \langle P_{k-1}\nabla h, \nabla h \rangle \equiv 0 \quad \text{on} \quad \Sigma.$$

The proof then finishes as that of Theorem 7.2. $\square$

8. The operator L_k acting on the function $\langle N, K \rangle$

In this section we will compute the operator L_k acting on the function $\langle N, K \rangle$. From (13) we see that

$$X(\langle N, K \rangle) = -\langle A(X), K \rangle = -\langle X, A(K^\top) \rangle$$

for every vector field $X \in \mathcal{X}(\Sigma)$, so that

$$\nabla \langle N, K \rangle = -A(K^\top).$$

Therefore,

(53) $\nabla_X(\nabla \langle N, K \rangle) = -\nabla_X(A(K^\top)) = -(\nabla_X A)(K^\top)) - A(\nabla_X K^\top),$

where $\nabla_X A$ denotes the covariant derivative of A,

$$(\nabla_X A)(Y) = \nabla A(Y, X) = \nabla_X(AY) - A(\nabla_X Y), \quad X, Y \in \mathcal{X}(\Sigma).$$

Recall now that the Codazzi equation of a hypersurface describes the normal component of $\overline{R}(X,Y)Z$, for tangent vector fields $X, Y, Z \in \mathcal{X}(\Sigma)$, in terms of the derivative of the shape operator. Specifically, the Codazzi equation of a spacelike hypersurface immersed into an arbitrary ambient spacetime is given by

$$\langle \overline{R}(X,Y)Z, N \rangle = \langle (\nabla_Y A)X - (\nabla_X A)Y, Z \rangle$$

or, equivalently,

$$(54) \qquad (\overline{R}(X,Y)N)^\top = (\nabla_X A)Y - (\nabla_Y A)X.$$

Then, using (14) and (54) into (53), we obtain that

$$\nabla_X(\nabla\langle N, K\rangle) = -(\nabla_{K^\top} A)(X)) - (\overline{R}(X, K^\top)N)^\top - f'(h)A(X) + \langle N, K\rangle A^2(X).$$

This implies

$$
\begin{aligned}
L_k(\langle N, K\rangle) &= -\mathrm{tr}((\nabla_{K^\top} A) \circ P_k) - \sum_{i=1}^n \langle \overline{R}(E_i, K^\top)N, P_k(E_i)\rangle \\
&\quad -f'(h)\mathrm{tr}(A \circ P_k) + \langle N, K\rangle \mathrm{tr}(A^2 \circ P_k) \\
(55) \qquad &= \binom{n}{k+1}\langle \nabla H_{k+1}, K\rangle - \sum_{i=1}^n \langle \overline{R}(E_i, K^\top)N, P_k(E_i)\rangle \\
&\quad +f'(h)c_k H_{k+1} + \binom{n}{k+1}\langle N, K\rangle(nH_1 H_{k+1} - (n-k-1)H_{k+2}),
\end{aligned}
$$

where $\{E_1, \ldots, E_n\}$ is an arbitrary local orthonormal frame on Σ. Here we are using the general fact that

$$\mathrm{tr}(\nabla_X A \circ P_k) = \mathrm{tr}(P_k \circ \nabla_X A) = -\binom{n}{k+1}\langle \nabla H_{k+1}, X\rangle$$

for every tangent vector field $X \in \mathcal{X}(\Sigma)$ (see equation (3.6) in [2]).

In order to write equation (55) in a more suitable way, observe that from (12) we can express

$$(56) \qquad \overline{R}(X, K^\top)N = \overline{R}(X, K)N - \langle N, K\rangle \overline{R}(N, X)N$$

for every tangent vector field $X \in \mathcal{X}(\Sigma)$. On the other hand, since $K^* = 0$, it follows from (33) that

$$\overline{R}(X, K)N = -\frac{f''(h)}{f(h)}\langle N, K\rangle X$$

for every tangent vector field $X \in \mathcal{X}(\Sigma)$, so that (56) becomes

$$\overline{R}(X, K^\top)N = -\langle N, K\rangle \left(\frac{f''(h)}{f(h)}X + \overline{R}(N, X)N\right).$$

Therefore,

$$
\begin{aligned}
\sum_{i=1}^n \langle \overline{R}(E_i, K^\top)N, P_k E_i\rangle &= -\langle N, K\rangle \left(\frac{f''(h)}{f(h)}\mathrm{tr}(P_k) + \mathrm{tr}(P_k \circ \overline{R}_N)\right) \\
(57) \qquad &= -\langle N, K\rangle \left(\frac{f''(h)}{f(h)}c_k H_k + \mathrm{tr}(P_k \circ \overline{R}_N)\right),
\end{aligned}
$$

where $\overline{R}_N : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ is the operator given by $\overline{R}_N(X) = (\overline{R}(N, X)N)^\top$. Using (57) in (55) we obtain the following result.

Lemma 8.1. *Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime $-I \times_f M^n$, with Gauss map N and height function h. Then, for every $k = 0, \ldots, n-1$ we have*

$$
\begin{aligned}
(58) \qquad L_k(\langle N, K \rangle) \;=\; & \binom{n}{k+1} \langle \nabla H_{k+1}, K \rangle + f'(h) c_k H_{k+1} \\
& + \binom{n}{k+1} \langle N, K \rangle (n H_1 H_{k+1} - (n-k-1) H_{k+2}) \\
& + \langle N, K \rangle \left(\operatorname{tr}(P_k \circ \overline{R}_N) + \frac{f''(h)}{f(h)} c_k H_k \right),
\end{aligned}
$$

where $\overline{R}_N : \mathcal{X}(\Sigma) \to \mathcal{X}(\Sigma)$ is the operator given by $\overline{R}_N(X) = (\overline{R}(N, X)N)^\top$.

Formula (58) becomes specially simple when $k = 0$. In that case $L_0 = \Delta$ is the Laplacian operator on Σ, and

$$
(59) \qquad\qquad \operatorname{tr}(P_0 \circ \overline{R}_N) = \operatorname{tr}(\overline{R}_N) = \overline{\operatorname{Ric}}(N, N),
$$

where $\overline{\operatorname{Ric}}$ is the Ricci tensor of $-I \times_f M^n$. Therefore, taking into account that

$$
(60) \qquad\qquad \langle N, \partial_t \rangle^2 = 1 + \|\nabla h\|^2,
$$

we obtain from (59) and (29) that

$$
(61) \qquad \operatorname{tr}(\overline{R}_N) + n\frac{f''(h)}{f(h)} = \operatorname{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2.
$$

Thus, when $k = 0$ expression (58) gives the following.

Corollary 8.2. *Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N and height function h. Then*

$$
\begin{aligned}
\Delta \langle N, K \rangle \;=\; & n \langle \nabla H, K \rangle + n f'(h) H + \langle N, K \rangle \|A\|^2 \\
(62) \qquad & + \langle N, K \rangle \left(\operatorname{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2 \right)
\end{aligned}
$$

For a proof, simply use (61) in (58) with $k = 0$ and observe that $n^2 H^2 - n(n-1)H_2 = \operatorname{tr}(A^2) = \|A\|^2$.

In order to find a better expression of (58) when $k \geq 1$, observe that from (33) we also get that

$$
\begin{aligned}
\overline{R}(N, X)N \;=\; & R_M(N^*, X^*)N^* - (((\log f)')^2(h) + (\log f)''(h)\langle N, \partial_t \rangle^2)X \\
& + (\log f)''(h)\langle X, \partial_t \rangle \partial_t + (\log f)''(h)\langle N, \partial_t \rangle\langle X, \partial_t \rangle N,
\end{aligned}
$$

for every tangent vector field $X \in \mathcal{X}(\Sigma)$. Using $\nabla h = -\partial_t^\top$ and (60), this implies

$$
\begin{aligned}
\overline{R}_N(X) \;=\; & (R_M(N^*, X^*)N^*)^\top + (\log f)''(h)\langle X, \nabla h \rangle \nabla h \\
& - (\frac{f''(h)}{f(h)} + (\log f)''(h)\|\nabla h\|^2)X.
\end{aligned}
$$

It follows from here that

$$
\begin{aligned}
\operatorname{tr}(P_k \circ \overline{R}_N) + \frac{f''(h)}{f(h)} c_k H_k \;=\; & \sum_{i=1}^n \langle R_M(N^*, E_i^*)N^*, P_k E_i \rangle \\
(63) \qquad & - (\log f)''(h)\left(c_k H_k \|\nabla h\|^2 - \langle P_k \nabla h, \nabla h \rangle \right),
\end{aligned}
$$

where $\{E_1, \ldots, E_n\}$ is an arbitrary local orthonormal frame on Σ.

Let us consider first the case where either $n = 2$ or $n \geq 3$ and M^n has constant sectional curvature. In both cases, we have that

$$(R_M(N^*, X^*)N^*)^\top = \kappa \langle N^*, N^* \rangle_M (X^*)^\top - \kappa \langle N^*, X^* \rangle_M (N^*)^\top,$$

where κ is either the (non-necessarily constant) Gaussian curvature of M^2 along the immersion ψ (when $n = 2$) or the constant sectional curvature of M^n (when $n \geq 3$). Using now the equalities (35), (37), (38) and (40), we obtain that

$$(R_M(N^*, X^*)N^*)^\top = \frac{\kappa}{f^2(h)} \left(\|\nabla h\|^2 X - \langle X, \nabla h \rangle \nabla h \right),$$

and (63) becomes

(64)
$$\operatorname{tr}(P_k \circ \overline{R}_N) + \frac{f''(h)}{f(h)} c_k H_k = \left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \left(c_k H_k \|\nabla h\|^2 - \langle P_k \nabla h, \nabla h \rangle \right).$$

Using this expression into (58), we obtain the following consequences.

Corollary 8.3. *Let $\psi : \Sigma^2 \to -I \times_f M^2$ be a spacelike surface immersed into a 3-dimensional GRW spacetime, with Gauss map N and height function h. Then,*

$$\begin{aligned}
L_1(\langle N, K \rangle) &= \langle \nabla H_2, K \rangle + 2f'(h)H_2 + \langle N, K \rangle 2 H_1 H_2 \\
&\quad - \langle N, K \rangle \left(\frac{K_M(\pi)}{f^2(h)} - (\log f)''(h) \right) \langle A \nabla h, \nabla h \rangle,
\end{aligned}$$

where $\pi : \Sigma^2 \to M^2$ denotes the projection of Σ^2 onto M^2, $\pi = \pi_M \circ \psi$.

Corollary 8.4. *Let Σ^n be a spacelike hypersurface immersed into a RW spacetime with Riemannian fiber M^n of constant sectional curvature κ, and let N and h denote its Gauss map and height function, respectively. Then, for every $k = 0, \ldots, n-1$ we have*

$$\begin{aligned}
L_k(\langle N, K \rangle) &= \binom{n}{k+1} \langle \nabla H_{k+1}, K \rangle + f'(h) c_k H_{k+1} \\
&\quad + \binom{n}{k+1} \langle N, K \rangle (n H_1 H_{k+1} - (n-k-1) H_{k+2}) \\
&\quad + \langle N, K \rangle \left(\frac{\kappa}{f^2(h)} - (\log f)''(h) \right) \left(c_k H_k \|\nabla h\|^2 - \langle P_k \nabla h, \nabla h \rangle \right).
\end{aligned}$$

To proceed on and obtain a better expression of (58) in the most general case, let us analyze the summatory term in (63). We will compute it by using a local orthonormal frame on Σ that diagonalizes A. It is worth pointing out that such a frame does not always exist; problems occur when the multiplicity of the principal curvatures changes. For that reason, we will work on the subset Σ_0 of Σ consisting of points at which the number of distinct principal curvatures is locally constant, which is an open dense subset of Σ [7, Paragraph 16.10]. Then, for every $p \in \Sigma_0$ there exists a local orthonormal frame defined on a neighbourhood of p that diagonalizes A and, hence, diagonalize P_k, that is, $\{E_1, \ldots, E_n\}$ such that $A(E_i) = \kappa_i E_i$ and $P_k E_i = \mu_{i,k} E_i$. Therefore, denoting by $K_M(N^* \wedge E_i^*)$ the sectional curvature in M^n of the 2-plane generated by N^* and E_i^*, we find that

$$\langle R_M(N^*, E_i^*)N^*, P_k E_i \rangle = \frac{\mu_{i,k}}{f^2(h)} K_M(N^* \wedge E_i^*) \|N^* \wedge E_i^*\|^2,$$

where we are using the fact that

$$\|N^* \wedge E_i^*\|^2 = f^4(h) \|N^* \wedge E_i^*\|_M^2.$$

Then we get the following.

Corollary 8.5. *Let Σ^n be a spacelike hypersurface immersed into a GRW spacetime, with Gauss map N and height function h. Then, for every $k = 0, \ldots, n-1$ we have*

$$
\begin{aligned}
(65) \quad L_k(\langle N, K \rangle) \;=\; & \binom{n}{k+1}\langle \nabla H_{k+1}, K \rangle + f'(h)c_k H_{k+1} \\
& + \binom{n}{k+1}\langle N, K \rangle(nH_1 H_{k+1} - (n-k-1)H_{k+2}) \\
& + \frac{\langle N, K \rangle}{f^2(h)} \sum_{i=1}^{n} \mu_{i,k} K_M(N^* \wedge E_i^*)\|N^* \wedge E_i^*\|^2 \\
& - \langle N, K \rangle(\log f)''(h)\left(c_k H_k \|\nabla h\|^2 - \langle P_k \nabla h, \nabla h \rangle\right).
\end{aligned}
$$

9. Umbilicity of hypersurfaces in GRW spacetimes

When $k = 0$, our formulas for the operator $L_0 = \Delta$ acting on the functions $g(h)$ and $\langle N, K \rangle$ allow us to re-obtain the uniqueness result given by Montiel in [28, Theorem 6] for hypersurfaces with constant mean curvature.

Theorem 9.1. *Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the null convergence condition, that is, satisfying*

$$
(66) \qquad\qquad \mathrm{Ric}_M \geq (n-1)\sup_I (f f'' - f'^2)\langle, \rangle_M,
$$

where Ric_M and $\langle, \rangle_M$ are respectively the Ricci and metric tensors of the compact Riemannian manifold M^n. Then the only compact spacelike hypersurfaces immersed into $-I \times_f M^n$ with constant mean curvature are the embedded slices $\{t\} \times M^n$, $t \in I$, unless in the case where $-I \times_f M^n$ is isometric to de Sitter spacetime in a neighborhood of Σ, which must be a round umbilical hypersphere. The latter case cannot occur if we assume that the inequality in (66) is strict.

Proof. Let us choose on Σ the future-pointing Gauss map N and let us consider the function $\phi = Hg(h) + \langle N, K \rangle \in \mathcal{C}^\infty(\Sigma)$. Since H is constant, from (18) and (62) we have that

$$
(67) \quad \Delta \phi = \langle N, K \rangle \left(\|A\|^2 - nH^2 + \mathrm{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2\right).
$$

Observe that, by Cauchy-Schwarz inequality, $\|A\|^2 - nH^2 \geq 0$ on Σ, with equality at umbilical points. Moreover, from (66) and (38) we also get

$$
\mathrm{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2 \geq 0.
$$

Therefore, since $\langle N, K \rangle < 0$ on Σ, by (67) we get that ϕ is superharmonic on Σ. But Σ being compact we have that ϕ must be constant. Hence $\Delta \phi = 0$, and $\|A\|^2 - nH^2$ vanishes on Σ, which means that Σ is totally umbilical, and

$$
(68) \qquad\qquad \mathrm{Ric}_M(N^*, N^*) - (n-1)(\log f)''(h)\|\nabla h\|^2 = 0 \quad \text{on} \quad \Sigma.
$$

If the inequality in (66) is strict, (68) is equivalent to $N^*(p) = 0$ at every $p \in \Sigma$, that is, $\|\nabla h(p)\| = 0$ at every $p \in \Sigma$. Then, h must be constant and Σ is a slice. In the general case, we obtain that Σ is totally umbilical and has constant mean curvature in $-I \times_f M^n$. Then, the result follows by observing that the case where the hypersurface is totally umbilical, but not a slice, can only occur if $-I \times_f M^n$ is

locally a de Sitter spacetime and Σ is a round umbilical hypersphere (see the classification of compact spacelike hypersurfaces with constant mean curvature given by Montiel in [28, Theorem 5]). $\qquad\square$

Our objective now is to extend the reasoning above to the case of hypersurfaces with constant higher order mean curvature, using our general formulas for the operators L_k acting on the functions $g(h)$ and $\langle N, K \rangle$. Specifically, we will prove the following uniqueness result, which extends Theorem 7.3 to the case of GRW ambient spacetimes. Here, instead of the null convergence condition we need to impose on $-I \times_f M^n$ the following stronger condition

$$(69) \qquad K_M \geq \sup_I (f f'' - f'^2),$$

where K_M stands for the sectional curvature of M^n. We will refer to (69) as the strong null convergence condition.

Theorem 9.2. *Let $-I \times_f M^n$ be a spatially closed GRW spacetime obeying the strong null convergence condition, with $n \geq 3$. Assume that Σ^n is a compact spacelike hypersurface immersed into $-I \times_f M^n$ which is contained in a slab $\Omega(t_1, t_2)$ on which f' does not vanish. If H_k is constant, with $2 \leq k \leq n$ then Σ is totally umbilical. Moreover, Σ must be a slice $\{t_0\} \times M^n$ (necessarily with $f'(t_0) \neq 0$), unless in the case where $-I \times_f M^n$ has positive constant sectional curvature and Σ is a round umbilical hypersphere. The latter case cannot occur if we assume that the inequality in (69) is strict.*

Proof. We may assume without loss of generality that $f'(h) > 0$ on Σ, and choose on Σ the future-pointing Gauss map N. Then, we know from Lemma 5.3 that there exists a point $p_0 \in \Sigma$ where all the principal curvatures (with respect to the chosen orientation) are negative. In particular, the constant H_k is positive and we can consider the function $\phi = H_k^{1/k} g(h) + \langle N, K \rangle \in \mathcal{C}^\infty(\Sigma)$. Since H_k is constant, from (18) and (65) we have that

$$(70) \qquad \begin{aligned} L_{k-1}\phi &= k\binom{n}{k}(H_k - H_k^{1/k}H_{k-1})f'(h) \\ &+ \binom{n}{k}\langle N, K \rangle (nH_1 H_k - (n-k)H_{k+1} - kH_k^{(k+1)/k}) \\ &+ \langle N, K \rangle \Theta, \end{aligned}$$

where

$$\begin{aligned} \Theta &= \frac{1}{f^2(h)} \sum_{i=1}^n \mu_{i,k-1} K_M(N^* \wedge E_i^*) \|N^* \wedge E_i^*\|^2 \\ &- (\log f)''(h) \left(c_{k-1} H_{k-1} \|\nabla h\|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle \right), \end{aligned}$$

and $P_{k-1} E_i = \mu_{i,k-1} E_i$ for every $i = 1, \ldots, n$.

Since Σ has an elliptic point, from the proofs of Theorem 7.3 and Theorem 7.4, respectively, we know that inequalities (49) and (52) hold on Σ for every k (inequality (52) was derived in the proof of Theorem 7.4, where we assumed that $k \leq n-1$; however (52) trivially holds also for $k = n$ since $H_{n+1} = 0$ by definition). From (49) we get that

$$(71) \qquad H_k - H_k^{1/k}H_{k-1} = H_k^{1/k}(H_k^{(k-1)/k} - H_{k-1}) \leq 0$$

on Σ, with equality if and only if Σ is totally umbilical. Using (52) we also get that

$$(72) \qquad nH_1H_k - (n-k)H_{k+1} - kH_k^{(k+1)/k} \geq kH_k(H_1 - H_k^{1/k}) \geq 0$$

on Σ. On the other hand, Lemma 3.2 (when $k = 2$) and Lemma 3.3 (when $k \geq 3$) apply here and imply that the operator L_{k-1} is elliptic or, equivalently, P_{k-1} is positive definite. In particular, its eigenvalues $\mu_{i,k-1}$ are all positive on Σ, and from (69) we have

$$(73) \qquad \mu_{i,k-1}K_M(N^* \wedge E_i^*)\|N^* \wedge E_i^*\|^2 \geq \mu_{i,k-1}\alpha\|N^* \wedge E_i^*\|^2$$

for every $i = 1, \ldots, n$, where we are writing $\alpha = \sup_I(ff'' - f'^2)$ for simplicity. From the decompositions

$$N = N^* - \langle N, \partial_t\rangle\partial_t, \quad E_i = E_i^* - \langle E_i, \partial_t\rangle\partial_t, \quad \text{and} \quad \partial_t = -\nabla h - \langle N, \partial_t\rangle N,$$

it follows easily that

$$\|N^* \wedge E_i^*\|^2 = \|\nabla h\|^2 - \langle E_i, \nabla h\rangle^2.$$

Therefore, (73) implies

$$
\begin{aligned}
\sum_{i=1}^n \mu_{i,k-1}K_M(N^* \wedge E_i^*)\|N^* \wedge E_i^*\|^2 &\geq \alpha\left(\text{tr}(P_{k-1})\|\nabla h\|^2 - \sum_{i=1}^n \mu_{i,k-1}\langle E_i, \nabla h\rangle^2\right) \\
&= \alpha\left(c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}(\nabla h), \nabla h\rangle\right),
\end{aligned}
$$

and then

$$(74) \qquad \Theta \geq \left(\frac{\alpha}{f^2(h)} - (\log f)''(h)\right)\left(c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}(\nabla h), \nabla h\rangle\right) \geq 0.$$

For the last inequality, observe that $\alpha/f^2(h) - (\log f)''(h) \geq 0$ holds trivially, and also observe that

$$c_{k-1}H_{k-1}\|\nabla h\|^2 - \langle P_{k-1}(\nabla h), \nabla h\rangle = \text{tr}(P_{k-1})\|\nabla h\|^2 - \langle P_{k-1}(\nabla h), \nabla h\rangle \geq 0$$

holds because P_{k-1} is positive definite.

Summing up, using (71), (72) and (74), and taking into account that $f'(h) > 0$ and $\langle N, K\rangle < 0$, we obtain from (70) that $L_{k-1}\phi \leq 0$ on Σ. Since L_{k-1} is an elliptic operator on the Riemannian manifold Σ, which is compact, we have, by the maximum principle, that ϕ must be constant. Hence, $L_{k-1}\phi = 0$ and the three terms in (70) vanish on Σ. In particular, (71) is an equality and Σ is a totally umbilical hypersurface. Since H_k is a positive constant and Σ is totally umbilical, we have that all the higher order mean curvatures are constant. In particular, H_1 is a positive constant and the result follows by Theorem 9.1. $\square$

References

1. J.A. Aledo, L.J. Alías and A. Romero, *Integral formulas for compact spacelike hypersurfaces in de Sitter space. Applications to the case of constant higher order mean curvature* J. Geom. Phys. **31** (1999), 195–208.

2. L.J. Alías, A. Brasil Jr. and A.G. Colares, *Integral formulae for spacelike hypersurfaces in conformally stationary spacetimes and applications*, Proc. Edinburgh Math. Soc. **46** (2003), 465–488.

3. L.J. Alías and S. Montiel, *Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes*. Differential geometry, Valencia, 2001, 59–69, World Sci. Publishing, River Edge, NJ, 2002.

4. L.J. Alías, A. Romero and M. Sánchez, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes*, Gen. Relativity Gravitation **27** (1995), 71–84.

5. L.J. Alías, A. Romero and M. Sánchez, *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems*, Tôhoku Math. J. **49** (1997), 337–345.

6. L.J. Alías, A. Romero and M. Sánchez, *Spacelike hypersurfaces of constant mean curvature in certain spacetimes*, Nonlinear Anal. **30** (1997), 655–661.

7. A.L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin 1987.

8. J.L. Barbosa and and A.G. Colares, *Stability of hypersurfaces with constant r-mean curvature*, Ann. Global Anal. Geom. **15** (1997), 277–297.

9. R. Bartnik, *Existence of maximal surfaces in asymptotically flat spacetimes*, Comm. Math. Phys. **94** (1984), 155–175.

10. R. Bartnik, P.T. Chrusciel and N. Ó Murchadha,*On maximal surfaces in asymptotically flat space-times*, Comm. Math. Phys. **130** (1990), 95–109.

11. R. Bartnik and L. Simon, *Spacelike hypersurfaces with prescribed boundary values and mean curvature*, Comm. Math. Phys. **87** (1982/83), 131–152.

12. D. Brill and F. Flaherty, *Isolated maximal surfaces in spacetime*, Comm. Math. Phys. **50** (1976), 157–165.

13. E. Calabi, *Examples of Bernstein problems for some nonlinear equations*. 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968) pp. 223–230 Amer. Math. Soc., Providence, R.I.

14. X. Cheng and H. Rosenberg, *Embedded positive constant r-mean cuvature hypersurfaces in $M^m \times R$*, An. Acad. Brasil. Ciênc. **77** (2005), 183–199.

15. S.Y. Cheng and S.T. Yau, *Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math. (2) **104** (1976), 407–419.

16. Y. Choquet-Bruhat, *Quelques propriétés des sousvariétés maximales d'une variété lorentzienne*, C. R. Acad. Sci. Paris Sér. A-B **281** (1975), A577–A580.

17. Y. Choquet-Bruhat, *Maximal submanifolds and submanifolds with constant mean extrinsic curvature of a Lorentzian manifold* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), 361–376.

18. Y. Choquet-Bruhat, A.E. Fischer and J.E. Marsden, *Maximal hypersurfaces and positivity of mass*, in Isolated Gravitating Systems in General Relativity. Proceedings of the International School of Physics "Enrico Fermi" (J. Ehlers, ed.), North-Holland Publishing Co., Amsterdam, 1979, p. 396-456

19. Y. Choquet-Bruhat and J.W. York Jr. *The Cauchy problem*, in General relativity and gravitation, Vol. 1, pp. 99–172, Plenum, New York-London, 1980.

20. F. Elbert, *Constant positive 2-mean curvature hypersurfaces*, Illinois J. Math. **46** (2002), 247–267.

21. L. Gàrding, *An inequality for hyperbolic polynomials*, J. Math. Mech. **8** (1959), 957–965.

22. C. Gerhardt, *H-surfaces in Lorentzian manifolds*, Comm. Math. Phys. **89** (1983), 523–553.

23. A.J. Goddard, *Some remarks on the existence of spacelike hypersurfaces of constant mean curvature*, Math. Proc. Cambridge Phil. Soc. **82** (1977), 489–495.

24. G. Hardy, J.E. Littlewood and G. Póyla, *Inequalities*, 2nd. edition, Cambridge Mathematical Library, Cambridge, 1989.

25. C.C. Hsiung, *Some integral formulas for closed hypersurfaces*, Math. Scand. **2** (1954), 286–294.

26. J.E. Marsden and F.J. Tipler, *Maximal hypersurfaces and foliations of constant mean curvature in general relativity*, Phys. Rep. **66** (1980), 109–139.

27. S. Montiel, *An integral inequality for compact spacelike hypersurfaces in de Sitter space and applications to the case of constant mean curvature*, Indiana Univ. Math. J. **37** (1988), 909–917.

28. S. Montiel, *Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes*, Math. Ann. **314** (1999), 529–553.

29. S. Montiel and A. Ros, *Compact hypersurfaces: The Alexandrov theorem for higher order mean curvatures*, in *Differential Geometry* (ed. B. Lawson and K. Tenenblat), pp. 279–296, Longman, Essex 1991

30. B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

31. R.C. Reilly, *Variational properties of functions of the mean curvature for hypersurfaces in space forms*, J. Differential Geom. **8** (1973), 465-477.

32. H. Rosenberg, *Hypersurfaces of constant curvature in space forms*, Bull. Sc. Math. **117** (1993), 211–239.

33. S. Stumbles, *Hypersurfaces of constant mean extrinsic curvature* Ann. Physics **133** (1981), 28–56.

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